“Principle of Indistinguishability” and equations of motion for particles with spin *

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Abstract

In this work we review the derivation of Dirac and Weinberg equations based on a “principle of indistinguishability” for the $(j,0)$ and $(0,j)$ irreducible representations (irreps) of the Homogeneous Lorentz Group (HLG). We generalize this principle and explore its consequences for other irreps containing $j \geq 1$. We rederive Ahluwalia-Kirchbach equation using this principle and conclude that it yields $\mathcal{O}(p^{2j})$ equations of motion for any representation containing spin j and lower spins. We also use the obtained generators of the HLG for a given representation to explore the possibility of the existence of first order equations for that representation. We show that, except for $j=\frac{1}{2}$, there exists no Dirac-like equation for the $(j,0)\oplus(0,j)$ representation nor for the $(\frac{1}{2},\frac{1}{2})$ representation. We rederive Kemmer-Duffin-Petieau (KDP) equation for the $(1,0)\oplus(\frac{1}{2},\frac{1}{2})\oplus(0,1)$ representation by this method and show that the $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ representation satisfies a Dirac-like equation which describes a multiplet of $j=\frac{3}{2}$ and $j=\frac{1}{2}$ with masses m and $\frac{m}{2}$ respectively.

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I. INTRODUCTION

The field theoretical description of interactions of particles with spin > 1 is a long standing problem. The interaction of a spin $\frac{3}{2}$ Rarita-Schwinger (RS) field minimally coupled to an external electromagnetic field was shown to be inconsistent more than fourthy years ago [1]. After this seminal work many authors have addressed this problem from different perspectives and for different interactions [2]. In particular the frequently posed requirement on unphysicality of the spin $\frac{1}{2}$ content of the RS field is the source of ambiguities in the description of the interactions of spin 3/2 particles with external fields (the so-called “off-shell” ambiguities [3]). The conclusion seems to be that it is not possible to construct a

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quantum theory of higher spin interacting particles and even some no-go theorems have been formulated [4] for the existence of massless particles with spin > 1. From a very phenomenological perspective this conclusion is very disappointing since on the one hand it closes the door for the possible existence of fundamental particles with $s > 1$ and on the other hand there exist a plethora of resonances with $s > 1$. Certainly, we know these are composite particles but in the long wavelength regime their composite nature is completely irrelevant and manifest only in the values of a few parameters (low energy constants). Thus, the long wave description of composite particles calls for a description for elementary systems with the any spin. This is particularly relevant for effective field theories of hadrons. In particular, the systematic expansion in powers of the momentum and quark masses for pseudoscalars [5], the so-called chiral perturbation theory (CHPT), is lost when we incorporate spin $\frac{1}{2}$ baryons [6]. Recovering a systematic expansion requires a heavy field expansion for spin $\frac{1}{2}$ degrees of freedom [7]. In this framework, the spin $\frac{3}{2}$ degrees of freedom have been shown to play a prominent role since they are not heavy enough to be "integrated out" and these degrees of freedom must be considered from the very start [8]. Ordinarily, they are treated within the frame work of the Rarita-Schwinger (RS) formalism. Clearly, one has to be careful when considering degrees of freedom with $s > 1$ because of fundamental quantum inconsistencies in the description of their interactions. However, as we showed in Ref. [9], the leading order of the heavy field expansion is free of the "off-shell" ambiguities and quantum inconsistencies. Beyond this order one cannot avoid problems when using the RS formalism for the description of spin $\frac{3}{2}$ degrees of freedom.

History has given plenty of examples that looking on phenomena from a perspective different but the originally accepted one, can lead to new and even surprising insights and in additon bring technical advantages. Historically the discovery of the equation of motion for spin $\frac{1}{2}$ particles by Dirac in 1931 was motivated by the desire to find a resolution of the problem of negative probabilities of the Klein-Gordon equation. After seven decades we are facing many different approaches to Dirac's equation. In one of the possibilities, it can be viewed just as a consequence of the transformation properties of the corresponding creation and annihilation operators under the Poincare Group [10,11] A different approach was put forward by Ryder in his well known textbook [12]. Ryder's method is based upon the representation theory of the Homogeneous Lorentz Group (HLG) in combination with certain identities valid only in the rest frame for which we here coined the term "*principle of indistinguishability*". It is that very principle on which we shall focus in the following.

In Ref. [13], Ryder's method was extended to incorporate discrete, C, P, and T symmetries into the wave equations for $(j,0)\oplus(0,j)$ by means of relative phases between $(j,0)$, and $(0,j)$. In particular Weinberg equations [10] have been shown to follow from specific choices for the corresponding phases. The obtained equations of motion are of the order $\mathcal{O}(p^{2j})$ in the momenta. Hence, for $s > 1$, the above procedure yields equations which are $\mathcal{O}(p^3)$ or higher order in p. The latter are known to possess acausal energy-momentum dispersion relations beyond the standard ones.
It is the goal of the present paper to generalize the "principle of indistinguishability" to representations containing $s > 1$ and other but $(j,0)\oplus(0,j)$.

We also constructed generators for a representation of interest from the generators of the simplest representations $(j,0)$ and $(0,j')$. Within this scheme we explore possibility for Dirac-like equations to exist for arbitrary spin.

II. HOMOGENEOUS LORENTZ GROUP AND PRINCIPLE OF INDISTINGUISHABILITY FOR THE $(J,0)\oplus(0,J)$ REPRESENTATIONS: BRIEF REVIEW

A. Irreducible representations

The local structure of the Homogeneous Lorentz group comprising boosts and rotations is expressed by means of the following commutators

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, \quad [J_i, K_j] = i\varepsilon_{ijk}K_k, \quad [K_i, K_j] = -i\varepsilon_{ijk}J_k. \tag{1}$$

The relations can be rewritten in terms of the new generators $\vec{A} = \frac{1}{2}\left(\vec{J} + i\vec{K}\right)$ and $\vec{B} = \frac{1}{2}\left(\vec{J} - i\vec{K}\right)$ which satisfy

$$[A_i, A_j] = i\varepsilon_{ijk}A_k, \quad [B_i, B_j] = i\varepsilon_{ijk}B_k, \quad [A_i, B_j] = 0. \tag{2}$$

Equations (2) shows that HLG is locally isomorphic to $SU(2)_A\otimes SU(2)_B$. This facilitates the classification of the irreps for this group which can be induced from those of $SU(2)$. Indeed, quantum states with well defined transformation properties under HLG can be classified according to two "angular momentum" labels (j,j') corresponding to the two subgroups spanned by the generators $\vec{A}$ and $\vec{B}$. Furthermore, under parity $\vec{J}\rightarrow\vec{J}$, $\vec{K}\rightarrow -\vec{K}$, thus $\vec{A}\rightarrow\vec{B}$, $\vec{B}\rightarrow\vec{A}$, hence the (j,j') and (j',j) representations are interchanged under parity. Under a Lorentz transformation spinors in the (j,j') representation (in momentum space) transform as

$$\Phi\left(p^{\mu}\right) = \Lambda\left(\dot{p}^{\mu}\rightarrow p^{\mu}\right)\Phi\left(\dot{p}^{\mu}\right) = e^{i(\vec{J}\cdot\vec{\vartheta}+\vec{K}\cdot\vec{\varphi})}\ \Phi\left(\dot{p}^{\mu}\right)\,, \tag{3}$$

where $\vec{\vartheta}, \vec{\varphi}$ denote the parameters of the transformation.

B. $(j,0)$ and $(0,j)$ representations.

The $(j,0)$ and $(0,j)$ are the simplest irreps of the HLG. For the $(j,0)$ representation we have $\vec{B}=0$, i.e. $\vec{J}=i\vec{K}$. For the $(0,j)$ representation $\vec{A}=0$ implies $\vec{J}=-i\vec{K}$. Following the literature we denote states belonging to the $(j,0)$ representation as "right" states and those in the $(0,j)$ representation as "left" states [1]. Under an HLG transformation, left and right states in momentum space transforms as

$$\Phi_R\left(p^{\mu}\right) = \Lambda_R\left(\dot{p}^{\mu}\rightarrow p^{\mu}\right)\Phi_R\left(\dot{p}^{\mu}\right) = e^{i\vec{J}\cdot(\vec{\vartheta}-i\vec{\varphi})}\ \Phi_R\left(\dot{p}^{\mu}\right), \tag{4}$$

[1] This nomenclature comes from the fact that in the massless case the "left" and "right" spinors turn out to be eigenstates of helicity operator $\vec{J}\cdot\hat{p}$ with eigenvalues $-s$ ("left-handed") and $+s$ ("right-handed") respectively. In the massive case however is just a book-keeping device for the site where the 0 is located in the $(j,0)$ or $(0,j)$ notation.

$$\Phi_L(p^\mu) = \Lambda_L(\dot{p}^\mu \to p^\mu)\,\Phi_L(\dot{p}^\mu) = e^{i\vec{J}\cdot(\vec{\vartheta}+i\vec{\varphi})}\;\Phi_L(\dot{p}^\mu)\,. \tag{5}$$

Pure Lorentz transformations (boosts) are obtained setting $\vec{\vartheta} = 0$ in(4,5)

$$\Phi_R(p^\mu) = B_R(\dot{p}^\mu \to p^\mu))\Phi_R(\dot{p}^\mu) = \exp(+\vec{J}\cdot\vec{\varphi})\Phi_R(\dot{p}^\mu)\,, \tag{6}$$

$$\Phi_L(p^\mu) = B_L(\dot{p}^\mu \to p^\mu)\,\Phi_L(\dot{p}^\mu) = \exp(-\vec{J}\cdot\vec{\varphi})\Phi_L(\dot{p}^\mu)\,. \tag{7}$$

In the case $\dot{p}^\mu = (m,\vec{0})$, where m stands for the mass of the corresponding particle (i.e. when $\dot{p}^\mu$ is the momentum in the rest frame of the particle), the angular momentum generators are just the spin operators. In this case, the parameters $\vec{\varphi}$ are related to the energy and momentum of the particle in the boosted frame as follows

$$\cosh\varphi = \gamma = \frac{E}{m} \quad \sinh\varphi = \beta\gamma = \frac{|\vec{p}|}{m} \quad \widehat{\varphi} = \frac{\vec{p}}{|\vec{p}|}. \tag{8}$$

Under rotations ($\vec{\varphi} = 0$ in (4,5)) the left and right states transform as

$$\Phi_R(p^\mu) = R_R(\dot{p}^\mu \to p^\mu))\Phi_R(\dot{p}^\mu) = \exp(i\vec{J}\cdot\vec{\vartheta})\Phi_R(\dot{p}^\mu)\;, \tag{9}$$

$$\Phi_L(p^\mu) = R_L(\dot{p}^\mu \to p^\mu)\,\Phi_L(\dot{p}^\mu) = \exp(i\vec{J}\cdot\vec{\vartheta})\Phi_L(\dot{p}^\mu)\;. \tag{10}$$

Notice that rest frame states in the $(j,0)$ representation have exactly the same transformation properties under rotations than those in the $(0,j)$ representation. Thus if particles are to be identified with the irreps of the HLG we are lead to the conclusion that there exist two kinds of states which are distinguished by their transformation properties under boosts but which when posed in the rest frame cannot be distinguished by their transformation properties under rotations. This is what we call the

Principle of Indistinguishability (PI)

At rest, a spinor belonging to the $(j,0)$ representation is indistinguishable from from a spinor in the $(0,j)$ representation [2]. Thus, the corresponding quantum states can differ at most by a phase [3]

$$\Phi_R(\vec{0}) = \varrho\;\Phi_L(\vec{0}), \qquad |\varrho| = 1. \tag{11}$$

[2] As far as I know this principle was first formulated for the case $j = \frac{1}{2}$ in the first edition of Ryder's book [12] although incomplete due to the missing of the phase ρ. This phase and its relation to intrinsic parity and anti-particle solutions to Dirac equation were firstly noticed in Refs. [13,14]. Further intriguing work on the consequences of relative phases between the building blocks of composite representations has been done in Refs. [13,15].

[3] Hereafter we will use the tri-momentum as the argument of spinors.

This principle can be used to rederive Dirac and Weinberg equations [4](see below). The question posed here is whether this principle can be generalized or not to other representations and used to derive new equations of motion (eom) for particles with $s \geq 1$. Alternatively it can give new insights into known eom's which could be rederived using this principle. The values of the phase ϱ in Eq.(11) can be restricted by imposing particular discrete spacetime properties onto the spinors of interest, like, say, parity covariance. Indeed, under parity the representations $(j,0)$ and $(0,j)$ are interchanged. At the quantum level this means that $\Pi\Phi_R(0) = \eta\Phi_L(0)$. Applying twice this operation and under the convention that we recover exactly the same state under two consecutive parity operations we obtain $\eta^2 = 1$. Transforming Eq.(11) under parity [5] we obtain

$$\Phi_L\left(\vec{0}\right) = \varrho\ \Phi_R\left(\vec{0}\right), \tag{12}$$

which when using Eq. (11) yield $\varrho^2 = 1$ i.e $\varrho = \pm 1$. Furthermore, if we require parity as a good symmetry, we are forced to consider a space representation comprising both $(j,0)$ and $(0,j)$. The natural space is $(j,0) \oplus (0,j)$ or its companion $(0,j) \oplus (j,0)$. The corresponding spinors are

$$\Psi_{RL}\left(\vec{p}\right) = \phi_R\left(\vec{p}\right) \oplus \phi_L\left(\vec{p}\right) = \begin{pmatrix} \Phi_R\left(\vec{p}\right) \\ \Phi_L\left(\vec{p}\right) \end{pmatrix}, \tag{13}$$

and

$$\Psi_{LR}\left(\vec{p}\right) = \phi_L\left(\vec{p}\right) \oplus \phi_R\left(\vec{p}\right) = \begin{pmatrix} \Phi_L\left(\vec{p}\right) \\ \Phi_R\left(\vec{p}\right) \end{pmatrix}, \tag{14}$$

These spinors transform under general Lorentz transformations as

$$\Lambda_{RL}\left(\vec{\theta},\vec{\varphi}\right)\Psi_{RL} = \begin{pmatrix} e^{i\vec{J}\cdot(\vec{\theta}-i\vec{\varphi})} & 0 \\ 0 & e^{i\vec{J}\cdot(\vec{\theta}+i\vec{\varphi})} \end{pmatrix} \begin{pmatrix} \Phi_R \\ \Phi_L \end{pmatrix}, \tag{15}$$

$$\Lambda_{LR}\left(\vec{\theta},\vec{\varphi}\right)\Psi_{LR} = \begin{pmatrix} e^{i\vec{J}\cdot(\vec{\theta}+i\vec{\varphi})} & 0 \\ 0 & e^{i\vec{J}\cdot(\vec{\theta}-i\vec{\varphi})} \end{pmatrix} \begin{pmatrix} \Phi_L \\ \Phi_R \end{pmatrix}. \tag{16}$$

Notice that under pure rotations the Ψ_{RL} and the Ψ_{LR} spinors transform identically which leads to a PI for the whole space which reads

$$\Psi_{RL}(\vec{0}) = \varrho\Psi_{LR}(\vec{0}) \text{ with } \varrho = \pm 1. \tag{17}$$

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[4]Actually the equations derived by Weinberg [10] missed the ϱ phase and are valid for the positive energy spinors only.

[5]Strictly speaking this is intrinsic parity, i.e parity in the rest frame. When acting on p-dependent spinors we must also transform $\vec{p} \to -\vec{p}$ which in the rest frame is trivial. We will call i-parity to this transformation in the following to distinguish it for the full parity transformation.

On the other hand, i-parity transformation for the fundamental representations $(j,0)$ and $(0,j)$ induces the following transformation for the $(j,0)\oplus(0,j)$ representation

$$\Psi_{RL}(\vec{p}) \rightarrow \Pi\Psi_{RL}(\vec{p}) = \eta\Psi_{LR}(\vec{p}) \quad \Rightarrow \quad \Pi = \eta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{18}$$

The factor $\eta = \pm 1$ is not relevant for the following derivation and will be omitted thorough this section. Using boost operators on the rest frame spinors

$$\Psi_{RL}(\vec{p}) = B_{RL}(\vec{p})\Psi_{RL}(\vec{0}), \tag{19}$$

with

$$B_{RL}(\vec{p}) = \begin{pmatrix} B_R(\vec{p}) & 0 \\ 0 & B_L(\vec{p}) \end{pmatrix} = \begin{pmatrix} e^{\vec{J}\cdot\vec{\varphi}} & 0 \\ 0 & e^{-\vec{J}\cdot\vec{\varphi}} \end{pmatrix}, \tag{20}$$

with similar relations for Ψ_{LR} and $B_{LR}(\vec{p})$. Notice that $B_{RL}(-\vec{p}) = B_{LR}(\vec{p})$. Furthermore, under i-parity

$$B_{RL}(\vec{p}) \rightarrow \Pi B_{RL}(\vec{p})\Pi = B_{LR}(\vec{p}) = B_{RL}(-\vec{p}). \tag{21}$$

Let us now boost the condition (17)

$$\begin{aligned}
\Psi_{RL}(\vec{p}) &= \varrho B_{RL}(\vec{p})\Psi_{LR}(0) = \varrho B_{RL}(\vec{p})\Pi\Psi_{RL}(0) \\
&= \varrho B_{RL}(\vec{p})\Pi B_{RL}^{-1}(\vec{p})\Psi_{RL}(\vec{p}) \\
&= \varrho B_{RL}(\vec{p})\Pi B_{RL}(-\vec{p})\Psi_{RL}(\vec{p}) \\
&= \varrho B_{RL}(\vec{p})\Pi^2 B_{RL}(\vec{p})\Pi\Psi_{RL}(\vec{p}) \\
&= \varrho B_{RL}^2(\vec{p})\Pi\Psi_{RL}(\vec{p}),
\end{aligned}$$

where we have used Eqs. (18,19,21) consecutively. This equation can be rewritten as

$$[B_{RL}^2(\vec{p})\Pi - \varrho]\Psi_{RL}(\vec{p}) = 0, \tag{22}$$

or explicitly in terms of the angular momentum generators [6]

$$\begin{pmatrix} -\varrho & e^{2\vec{J}\cdot\vec{\varphi}} \\ e^{-2\vec{J}\cdot\vec{\varphi}} & -\varrho \end{pmatrix}\Psi_{RL}(\vec{p}) = 0. \tag{23}$$

This is the equation of motion that any field in the $(j,0)\oplus(0,j)$ representation must satisfy. The explicit form of these equations in terms of momentum operators require to evaluate the exponentials. The explicit form - for arbitrary j - of these exponentials can be obtained from the general relations for the hyperbolic functions given in appendix A of Ref. [10]. As an example, let us consider the cases $j = \frac{1}{2}, 1, \frac{3}{2}$.

[6] If we keep the η phase thorough, the ϱ phase in Eq.(23) must be replaced by $\eta\varrho = \pm 1$.

C. $\left(j=\frac{1}{2}\right)$: Dirac Equation.

In the case $j=\frac{1}{2}$, $J=\frac{\vec{\sigma}}{2}$ satisfy the *bilinear* algebra $\{J_i,J_j\}=\frac{1}{2}\delta_{ij}$ which can be used to evaluate the exponential as

$$e^{\vec{\sigma}\cdot\vec{\varphi}}= \ \cosh\varphi\,\mathbf{1}\ +\ (\vec{\sigma}\cdot\widehat{p})\sinh\varphi. \tag{24}$$

From Eqs. (8,23,24) we obtain

$$\begin{pmatrix} -1 & \varrho\,\frac{E+\vec{\sigma}\cdot\vec{p}}{m} \\ \varrho\,\frac{E-\vec{\sigma}\cdot\vec{p}}{m} & -1 \end{pmatrix}\Psi\left(\vec{p}\right)=0. \tag{25}$$

In terms of the 4×4 matrices

$$\gamma^0=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\qquad \gamma^i=\begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}. \tag{26}$$

We can rewrite Eq. (25) in Dirac's form

$$\left(\gamma^\mu p_\mu-\varrho\ m\right)\Psi\left(\vec{p}\right)=0. \tag{27}$$

This is the conventional Dirac theory where $\varrho=1$ corresponds to positive energy spinors and $\varrho=-1$ to negative energy spinors. When we switch to Dirac representation for the γ matrices i-parity operator is diagonal and the corresponding spinors have opposite intrinsic parity. The particular assignment of intrinsic parity depends on the choice for the phase η which remains arbitrary. It is worth noticing that γ_μ satisfy Dirac algebra $\{\gamma^\alpha,\gamma^\beta\}=2g^{\alpha\beta}$ *which is just the covariant version of the algebra satisfied by* $\frac{J_i}{s}$.

D. $(j=1)$.

In the case $j=1$, the angular momentum operators, in addition to the Lie algebra of $SU(2)$, can be shown to satisfy also the *trilinear algebra*

$$J_iJ_jJ_k+J_kJ_jJ_i=\delta_{ij}J_k+\delta_{jk}J_i, \tag{28}$$

which can be used to calculate the exponentials in Eq.(23). Then, from Eq.(23) we obtain

$$\begin{pmatrix} -\varrho m^2 & m^2+2E\left(\vec{J}\cdot\vec{p}\right)\ +2\left(\vec{J}\cdot\vec{p}\right)^2 \\ m^2-2E\left(\vec{J}\cdot\vec{p}\right)+2\left(\vec{J}\cdot\vec{p}\right)^2 & -\varrho m^2 \end{pmatrix}\begin{pmatrix} \Phi_R\left(\vec{p}\right) \\ \Phi_L\left(\vec{p}\right) \end{pmatrix}=0. \tag{29}$$

Defining now the matrices

$$\gamma_{00}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \gamma_{0i}=\gamma_{i0}=\begin{pmatrix} 0 & J_i \\ -J_i & 0 \end{pmatrix},\quad \gamma_{ij}=g_{ij}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}+\begin{pmatrix} 0 & \{J_i,J_j\} \\ \{J_i,J_j\} & 0 \end{pmatrix}, \tag{30}$$

Eq.(29) can be cast into Weinberg's form (up to the phase ϱ which does not appear in Weinberg equations)

$$\left[\gamma_{\mu\nu}p^\mu p^\nu-\varrho\ m^2\right]\Psi\left(\vec{p}\right)=0. \tag{31}$$

E. $\left(j = \frac{3}{2}\right)$.

The generators for rotations in the case of $j = \frac{3}{2}$ can be shown to satisfy the *cuatrilinear algebra*

$$\{J_{ij}, J_{kl}\} = 5(\delta_{ij} J_{kl} + \delta_{kl} J_{ij}) - \frac{9}{2}\delta_{ij}\delta_{kl}. \tag{32}$$

where $J_{ij} \equiv \{J_i, J_j\}$. Similar calculations and defining the totaly symmetric matrices

$$\gamma_{000} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_{00i} = \begin{pmatrix} 0 & \frac{2}{3}J_i \\ -\frac{2}{3}J_i & 0 \end{pmatrix}, \quad \gamma_{0ij} = \begin{pmatrix} 0 & \frac{1}{4}g_{ij} + \frac{1}{6}\{J_i, J_j\} \\ \frac{1}{4}g_{ij} + \frac{1}{6}\{J_i, J_j\} & 0 \end{pmatrix}, \tag{33}$$

$$\gamma_{ijk} = \begin{pmatrix} 0 & \frac{4}{3}J_{ijk} + \frac{7}{3}T_{ijk} \\ -\frac{4}{3}J_{ijk} - \frac{7}{3}T_{ijk} & 0 \end{pmatrix}, \tag{34}$$

where

$$J_{ijk} = \frac{1}{6}\left(J_i\{J_j, J_k\} + J_j\{J_k, J_i\} + J_k\{J_i, J_j\}\right), \quad T_{ijk} = \frac{1}{3}\left(g_{ij}J_k + g_{jk}J_i + g_{ki}J_j\right), \tag{35}$$

yield the equation of motion

$$\left[\gamma_{\mu\nu\sigma}p^\mu p^\nu p^\sigma - \varrho m^3\right]\Psi(\vec{p}) = 0, \tag{36}$$

which is an $\mathcal{O}(p^3)$ equation. These results can be easily generalized. For general j, the angular momentum operators, in addition to satisfy $SU(2)$ Lie algebra will also satisfy a $2j+1$-linear algebra which will yield

$$e^{2\vec{J}\cdot\varphi} \approx \mathcal{O}(p^{2j}), \tag{37}$$

and fields transforming in the $(j,0) \oplus (0,j)$ representation will obey equations of the form

$$\left[\gamma^{\mu_1\ldots\mu_{2j}}p_{\mu_1}\ldots p_{\mu_{2j}} - \varrho m^{2j}\right]\Psi(\vec{p}) = 0. \tag{38}$$

i.e. equations of $\mathcal{O}(p^{2j})$ which are intractable for $j > 1$.

III. GENERALIZATION TO DIRECT PRODUCTS.

A. $(j_1, 0) \otimes (0, j_2)$ representation: Generalities.

The simplest representations beyond $(j,0) \oplus (0,j)$ are those obtained as tensor product of the $(j_1, 0)$ and $(0, j_2)$ representations, i.e., the $(j_1, 0) \otimes (0, j_2)$ representations. The corresponding rotation operators are given by

$$R(\vec{\vartheta}) = R_R(\vec{\vartheta}) \otimes R_L(\vec{\vartheta}) \equiv R_{RL}(\vec{\vartheta}) = e^{i\vec{J}_1\cdot\vec{\vartheta}} \otimes e^{i\vec{J}_2\cdot\vec{\vartheta}}, \tag{39}$$

whereas for boosts we obtain

$$B(\vec{\varphi}) = B_R(\vec{\varphi}) \otimes B_L(\vec{\varphi}) \equiv B_{RL}(\vec{\varphi}) = e^{\vec{J}_1\cdot\vec{\varphi}} \otimes e^{-\vec{J}_2\cdot\vec{\varphi}}, \tag{40}$$

The generators, in the tensor product basis (TPB), for these transformations satisfy (for arbitrary $\widehat{n}$)

$$\vec{J}\cdot\widehat{n} = \frac{1}{i}\frac{\partial R_R(\vartheta)}{\partial\vartheta} = \vec{J}_1\cdot\widehat{n}\otimes \mathbf{1}_{(2j_2+1)\times(2j_2+1)} + \mathbf{1}_{(2j_1+1)\times(2j_1+1)}\otimes \vec{J}_2\cdot\widehat{n}, \tag{41}$$

$$\vec{K}\cdot\widehat{n} = \frac{1}{i}\frac{\partial B_R(\varphi)}{\partial\varphi} = \frac{1}{i}\vec{J}_1\cdot\widehat{n}\otimes \mathbf{1}_{(2j_2+1)\times(2j_2+1)} - \mathbf{1}_{(2j_1+1)\times(2j_1+1)}\otimes \vec{J}_2\cdot\widehat{n}, \tag{42}$$

thus the generators for the $(j_1,0)\otimes(0,j_2)$ representation, in the TPB are

$$\vec{J} = \vec{J}_1\otimes\mathbf{1} + \mathbf{1}\otimes\vec{J}_2, \quad i\vec{K} = \vec{J}_1\otimes\mathbf{1} - \mathbf{1}\otimes\vec{J}_2, \tag{43}$$

where we omitted the dimensionality of the unit matrices which can be easily traced.

Under i-parity the $(j_1,0)\otimes(0,j_2)$ representations are mapped onto the $(0,j_1)\otimes(j_2,0)$ representations which are unitarily equivalent to $(j_2,0)\otimes(0,j_1)$ (by unitarily equivalent here we mean the existence of a unitary transformation which connects these representations in the rest frame). Thus the construction of a parity invariant theory forces us to consider the $[(j_1,0)\otimes(0,j_2)]\oplus[(0,j_1)\otimes(j_2,0)]$ or $[(j_1,0)\otimes(0,j_2)]\oplus[(j_2,0)\otimes(0,j_1)]$ as our representation space, except in the case of $j_1=j_2$ where $(j_1,0)\otimes(0,j_1)$ spans an irreducible representation for the Parity operator. Indeed, under parity $(j_1,0)\otimes(0,j_1)$ goes into $(0,j_1)\otimes(j_1,0)$ which is unitarily equivalent to $(j_1,0)\otimes(0,j_1)$.

The construction of the equations of motion for fields transforming as $[(j_1,0)\otimes(0,j_2)]$ is more transparent when we work with irreducible representations with respect to the rotations subgroup. Thus we need to pass from the tensor product basis (TPB) $|j_1,m_1\rangle\otimes|j_2,m_2\rangle$ to the total angular momentum basis (TAMB) $|j_1j_2;j,m\rangle$. The unitary transformation U connecting these basis is the matrix whose elements are the corresponding Clebsch-Gordon coefficients ($\langle j_1m_1;j_2m_2|j_1j_2;jm\rangle$).

Let us start with the explicit construction with the simplest case $j_1=j_2=\frac{1}{2}$.

B. $(\frac{1}{2},0)\otimes(0,\frac{1}{2})$: Proca representation.

The states corresponding to $(\frac{1}{2},0)\otimes(0,\frac{1}{2})$ are $|\frac{1}{2}m\rangle_R\otimes|\frac{1}{2}m'\rangle_L$ whereas those belonging to $(0,\frac{1}{2})\otimes(\frac{1}{2},0)$ are $|\frac{1}{2}m\rangle_L\otimes|\frac{1}{2}m'\rangle_R$. Explicitly the spinors are given by

$$\phi_{RL}\equiv\phi_R\otimes\phi_L = \ l.c.\ \begin{pmatrix} |+\rangle_R|+\rangle_L \\ |+\rangle_R|-\rangle_L \\ |-\rangle_R|+\rangle_L \\ |-\rangle_R|-\rangle_L \end{pmatrix}, \quad \phi_{LR}\equiv\phi_L\otimes\phi_R = \ l.c.\ \begin{pmatrix} |+\rangle_L|+\rangle_R \\ |+\rangle_L|-\rangle_R \\ |-\rangle_L|+\rangle_R \\ |-\rangle_L|-\rangle_R \end{pmatrix}. \tag{44}$$

where $l.c.$ stands for linear combination. In the latter equations we used the customary notation $|\frac{1}{2}\frac{1}{2}\rangle\equiv|+\rangle$ and $|\frac{1}{2}-\frac{1}{2}\rangle\equiv|-\rangle$, and the L,R subindices to denote their different transformation properties under boosts. Notice that $\phi_{LR}=U\ \phi_{RL}$ with

$$U \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{45}$$

Under i-parity

$$\phi_{RL} \longrightarrow \eta\, \phi_{LR} = \eta\, U \phi_{RL} \Rightarrow \Pi = \eta\, U\,, \tag{46}$$

where η is a phase which is restricted to $\eta = \pm 1$ because $\Pi^2 = 1$. The boost and rotation operators can be constructed as

$$R_{RL}(\vec{\theta}) = e^{i\frac{\vec{\sigma}}{2}\cdot\vec{\theta}} \otimes e^{i\frac{\vec{\sigma}}{2}\cdot\vec{\theta}} = R_{LR}(\vec{\theta}), \quad B_{RL}(\vec{\varphi}) = e^{\frac{\vec{\sigma}}{2}\cdot\vec{\varphi}} \otimes e^{-\frac{\vec{\sigma}}{2}\cdot\vec{\varphi}}, \quad B_{LR}(\vec{\varphi}) = e^{-\frac{\vec{\sigma}}{2}\cdot\vec{\varphi}} \otimes e^{\frac{\vec{\sigma}}{2}\cdot\vec{\varphi}}. \tag{47}$$

The corresponding generators read

$$\vec{J}_{RL} = \frac{1}{2}(\vec{\sigma} \otimes \mathbf{1} + \mathbf{1} \otimes \vec{\sigma}) = \vec{J}_{LR}\,, \qquad i\vec{K}_{RL} = \frac{1}{2}(\vec{\sigma} \otimes \mathbf{1} - \mathbf{1} \otimes \vec{\sigma}) = -i\vec{K}_{LR}\,. \tag{48}$$

In terms of E and $\vec{p}$ the boost operators for rest frame states read

$$B_{RL}(\vec{p}) = \frac{1}{2m(E+m)}[E + m + \vec{\sigma}\cdot\vec{p}] \otimes [E + m - \vec{\sigma}\cdot\vec{p}] = B_{LR}(-\vec{p})\,. \tag{49}$$

It can be shown that under i-parity

$$B_{RL}(\vec{p}) \longrightarrow \Pi B_{RL}(\vec{p}) \Pi = B_{RL}(-\vec{p}) = B_{LR}(\vec{p})\,. \tag{50}$$

Notice that the PI $\phi_R(0) = \varrho\phi_L(0)$ and its i-parity transformed $\phi_L(0) = \varrho\phi_R(0)$ induces the following principle for the composed representation

$$\phi_{RL}(0) = \phi_{LR}(0)\,. \tag{51}$$

Boosting this equation and a little of algebra yields

$$\begin{aligned} \phi_{RL}(\vec{p}) &= B_{RL}(\vec{p})\phi_{LR}(0) = B_{RL}(\vec{p})\Pi\phi_{RL}(0) \\ &= B_{RL}(\vec{p})\Pi B^{-1}_{RL}(\vec{p})\phi_{RL}(\vec{p}) \\ &= B_{RL}(\vec{p})\Pi B_{RL}(-\vec{p})\phi_{RL}(\vec{p}) \\ &= B_{RL}(\vec{p})\Pi^2 B_{RL}(\vec{p})\Pi\phi_{RL}(\vec{p}) \\ &= B^2_{RL}(\vec{p})\Pi\phi_{RL}(\vec{p}). \end{aligned}$$

Thus, the PI yields the following equation of motion in the TPB.

$$[B^2_{RL}(\vec{p})U - \eta]\phi_{RL}(\vec{p}) = 0. \tag{52}$$

Although we derived this equation for the case $j = \frac{1}{2}$ it is clear that it holds for any j with the appropriate change in the generators of rotations. For $j = \frac{1}{2}$ the squared boost operator satisfy

$$B^2_{RL}(\vec{p}) = B^2_R(\vec{p}) \otimes B^2_L(\vec{p}) = \frac{1}{m^2}[E+\vec{\sigma}\cdot\vec{p}] \otimes [E-\vec{\sigma}\cdot\vec{p}]\,, \tag{53}$$

and the equation of motion for the Proca representation, in the TPB, reads

$$\Big([E+\vec{\sigma}\cdot\vec{p}] \otimes [E-\vec{\sigma}\cdot\vec{p}]\ U \ -\eta m^2\Big)\,\phi_{RL}(\vec{p}) = 0. \tag{54}$$

It is interesting to write this equation in the TAMB. This basis is related to the TPB as $|TAMB\rangle = M_{cb}|TPB\rangle$. Explicitly

$$\begin{pmatrix} |0,0\rangle \\ |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} |+\rangle_R|+\rangle_L \\ |+\rangle_R|-\rangle_L \\ |-\rangle_R|+\rangle_L \\ |-\rangle_R|-\rangle_L \end{pmatrix}. \tag{55}$$

Under this change of basis:

$$\Pi \longrightarrow \tilde{\Pi} = M_{cb}\Pi M^\dagger_{cb} = \eta Diag(-1,1,1,1)\,, \tag{56}$$

$$\vec{J}^{TPB}_{RL} \longrightarrow \vec{J}^{TAMB}_{RL} = \begin{pmatrix} \vec{0}_{1\times1} & \vec{0}_{1\times3} \\ \vec{0}_{3\times1} & \vec{L} \end{pmatrix}, \quad i\vec{K}^{TPB}_{RL} \longrightarrow i\vec{K}^{TAMB}_{RL} = \begin{pmatrix} \vec{0}_{1\times1} & \vec{B}^\dagger \\ \vec{B} & \vec{0}_{3\times3} \end{pmatrix}, \tag{57}$$

where

$$B^\dagger_1 = \frac{1}{\sqrt{2}}(-1,0,1), \quad B^\dagger_2 = -\frac{i}{\sqrt{2}}(1,0,1), \quad B^\dagger_3 = (0,1,0)\,, \tag{58}$$

and $\vec{L}$ denote the angular momentum operators for spin 1. These relations make clear that the field $\tilde{\phi}_{RL} \equiv M_{cb}\phi_{RL}$ describes a multiplet composed of a spin zero field and a spin 1 field with opposite intrinsic parities. The choice $\eta = -1$ yields $(\tilde{\Pi})_{\mu\nu} = g_{\mu\nu}$ and with this choice $\tilde{\phi}_{RL}$ describes a multiplet of a spin 0 field with positive intrinsic parity and a spin 1 field with with negative intrinsic parity.

Transforming Eq.(54) to the TAMB we obtain

$$[\tilde{B}^2_{RL}(\vec{p})\tilde{U} + \eta]\tilde{\phi}_{RL}(\vec{p}) = 0\,, \tag{59}$$

which yields

$$[\mathcal{O} - \eta m^2]\tilde{\phi}_{RL}(\vec{p}) = 0\,, \tag{60}$$

where

$$\mathcal{O} = \begin{pmatrix} E^2+\vec{p}^2 & \sqrt{2}Ep_+ & -2Ep_z & -\sqrt{2}Ep_- \\ -\sqrt{2}Ep_- & -(E^2-p_z^2) & \sqrt{2}p_zp_+ & p_-^2 \\ 2Ep_z & \sqrt{2}p_zp_+ & -E^2-p_z^2+p_+p_- & -\sqrt{2}p_zp_- \\ \sqrt{2}Ep_+ & p_+^2 & \sqrt{2}p_zp_+ & -(E^2-p_z^2) \end{pmatrix}, \tag{61}$$

or

$$(p^\mu p^\nu \Lambda_{\mu\nu} - \eta m^2)\tilde{\phi}(\vec{p}) = 0\,, \tag{62}$$

with

$$\Lambda_{\mu\nu} = \Lambda_{\nu\mu}, \quad \Lambda_{00} = Diag(1,-1,-1,-1) \;\; \text{etc.} \tag{63}$$

This equation was first derived in Ref. [18] using projectors techniques instead of a PI. In that work different (but equivalent) representation for the $\Lambda_{\mu\nu}$ matrices are obtained. An exhaustive analysis of the properties of this equation can also be found in Ref. [18]. Here we just stress that this equation follows from the same principle as the Dirac and Weinberg equations.

C. $(1,0)\oplus(\frac{1}{2},\frac{1}{2})\oplus(0,1)$: Kemmer-Duffin-Petieau representation.

Historically some other formalisms for the description of particles with spin one have been considered. In particular the Kemmer-Duffin-Petieau (KDP) formalism [20,21] which uses the $(1,0)\oplus(\frac{1}{2},\frac{1}{2})\oplus(0,1)$ representation of the Lorentz group. Let us explore the possible consequences of the PI for this representation.

The corresponding analysis is straightforward if we use the representation $(1,0))\oplus(0,1)\oplus(\frac{1}{2},\frac{1}{2})$ which is connected to the KDP representation by an obvious unitary transformation and will also be called KDP representation in the following. The corresponding spinors have the following structure

$$\phi^{KDP}_{RL} = \begin{pmatrix} \phi^W_{RL} \\ \phi^P_{RL} \end{pmatrix}, \tag{64}$$

where ϕ^W_{RL} and ϕ^P_{RL} denote spinors in the $(1,0))\oplus(0,1)$ (Weinberg representation in the following) and Proca representations respectively. Under i-parity

$$\begin{aligned} \phi^{KDP}_{RL} = \phi^W_{RL}\oplus\phi^P_{RL} \longrightarrow \phi^{KDP}_{LR} &= \phi^W_{LR}\oplus\phi^P_{LR} = (\Pi^W\phi^W_{LR})\oplus(\Pi^P\phi^P_{LR}) \\ &= \Pi^{KDP}\phi^{KDP}_{LR}, \end{aligned} \tag{65}$$

thus i-parity operator has the following structure

$$\Pi^{KDP} \equiv \Pi^W \oplus \Pi^P = \begin{pmatrix} \Pi^W & 0 \\ 0 & \Pi^P \end{pmatrix}. \tag{66}$$

Clearly, there is no mixing between ϕ^W_{RL} and ϕ^P_{RL} under parity or Lorentz transformations. The PI for this representation reads

$$\phi^{KDP}_{RL}(0) = \;\tilde{\varrho}\phi^{KDP}_{LR}(0)\,, \tag{67}$$

where now $\tilde{\varrho}$ stands for the block-diagonal matrix $Diag(\varrho\mathbf{1}_{6\times 6}, \eta\mathbf{1}_{4\times 4})$. Boosting this equation we obtain

$$[B^{KDP}(\vec{p})^2\Pi^{KDP} - \tilde{\varrho}]\;\phi^{KDP}_{RL}(\vec{p}) = 0\,, \tag{68}$$

where $B^{KDP}(\vec{p})^2 = B^W(\vec{p})^2 \oplus B^P(\vec{p})^2$, i.e.

$$\left[\begin{pmatrix} B^W(\vec{p})^2\Pi^W - \varrho & 0 \\ 0 & B^P(\vec{p})^2\Pi^P - \eta \end{pmatrix}\right] \begin{pmatrix} \phi^W_{RL}(\vec{p}) \\ \phi^P_{RL}(\vec{p}) \end{pmatrix} = 0. \tag{69}$$

As a final result the equation splits into two independent equations, one for the Weinberg field and another one for the Proca field with a common mass.

D. $(\frac{1}{2},\frac{1}{2}) \otimes [(\frac{1}{2},0) \oplus (0,\frac{1}{2})]$: Rarita-Schwinger representation.

Let us now study the Rarita- Schwinger representation in the light of the PI. The corresponding spinors are constructed as a direct product of the Dirac and Proca fields. Under i-parity the RS field transforms as follows

$$\phi^{RS}_{RL} = \phi^P_{RL} \otimes \phi^D_{RL} \longrightarrow \phi^{RS}_{LR} = \phi^P_{LR} \otimes \phi^D_{LR} = (\Pi^P \phi^P_{LR}) \otimes (\Pi^D \phi^D_{LR}) = \Pi^{RS}\phi^{RS}_{LR}. \tag{70}$$

with $\Pi^{RS} \equiv \Pi^P \otimes \Pi^D$. Rotation operators for the $RL \equiv \left[(\frac{1}{2},0) \otimes (0,\frac{1}{2})\right] \otimes \left[(\frac{1}{2},0) \oplus (0,\frac{1}{2})\right]$ and the $LR \equiv \left[(0,\frac{1}{2}) \otimes (\frac{1}{2},0)\right] \otimes \left[(0,\frac{1}{2}) \oplus (\frac{1}{2},0)\right]$ representations are

$$R^{RS}_{RL}(\vec{\vartheta}) = R^P_{RL}(\vec{\vartheta}) \otimes R^D_{RL}(\vec{\vartheta}), \qquad R^{RS}_{LR}(\vec{\vartheta}) = R^P_{LR}(\vec{\vartheta}) \otimes R^D_{LR}(\vec{\vartheta}). \tag{71}$$

The identical transformation properties under rotations for the Dirac and Proca fields induces a PI for ϕ^{RS}_{LR} and ϕ^{RS}_{RL} in the rest frame. Boosts operators for the RL representation reads

$$B^{RS}_{RL}(\vec{\varphi}) = B^P_{RL}(\vec{\varphi}) \otimes B^D_{RL}(\vec{\varphi}), \tag{72}$$

which under i-parity transforms as

$$\Pi^{RS} B^{RS}_{RL}(\vec{p}) \Pi^{RS} = B^{RS}_{RL}(-\vec{p}). \tag{73}$$

The PI for this representation

$$\phi^{RS}_{RL}(0) = \varrho \phi^{RS}_{LR}(0), \tag{74}$$

together with (73) yields

$$[B^{RS}(\vec{p})^2 \Pi^{RS} - \varrho]\ \phi^{RS}_{RL}(\vec{p}) = 0. \tag{75}$$

Notice that

$$B^{RS}(\vec{p})^2 = B^P(\vec{p})^2 \otimes B^D(\vec{p})^2 = \mathcal{O}(p^2) \otimes \mathcal{O}(p) = \mathcal{O}(p^3), \tag{76}$$

and the PI yields an $\mathcal{O}(p^3)$ equation which we will not push further here due to the known acausalities for any $\mathcal{O}(p^3)$ equation.

E. Other representations containing spin $\frac{3}{2}$

To study other possibilities such as $(1,0)\otimes(0,\frac{1}{2})$ we need to consider in general the structure for the $(j_1,0)\otimes(0,j_2)$ representation with $j_1\neq j_2$. Under parity $(j_1,0)\otimes(0,j_2)\rightarrow(0,j_1)\otimes(j_2,0)$. A theory for quantum fields which consider parity as a good symmetry would require to consider the whole $[(j_1,0)\otimes(0,j_2)]\oplus[(0,j_1)\otimes(j_2,0)]$ space. We denote $(j_1,0)\otimes(0,j_2)$ as "left" representation and $(0,j_1)\otimes(j_2,0)$ as "right"representation thorough this section. In the TPB the generators for these representations read

$$\vec{J}_L=\vec{J}_1\otimes\mathbf{1}+\mathbf{1}\otimes\vec{J}_2,\quad i\vec{K}_L=-\vec{J}_1\otimes\mathbf{1}+\mathbf{1}\otimes\vec{J}_2. \tag{77}$$

$$\vec{J}_R=\vec{J}_1\otimes\mathbf{1}+\mathbf{1}\otimes\vec{J}_2,\quad i\vec{K}_R=\vec{J}_1\otimes\mathbf{1}-\mathbf{1}\otimes\vec{J}_2. \tag{78}$$

Notice that the relations $J_L=J_R,\ K_L=-K_R$, which are valid for the $(j,0)$ and $(0,j)$ representations, are also valid in this case. Hence, in the case at hand, when posed in the rest frame, it is also impossible to distinguish fields transforming in the "left" from those transforming in the "right" representation which can be used to derive the corresponding equation of motion. As for the parity operator it has in TPB a simple form that follows from the transformation properties of the "left" and "right" representations. There, the generators for rotations and boosts for the whole $[(j_1,0)\otimes(0,j_2)]\oplus[(0,j_1)\otimes(j_2,0)]$ representation have a simple block-diagonal form

$$\vec{J}=\begin{pmatrix}\vec{J}_R & 0\\ 0 & \vec{J}_L\end{pmatrix},\quad \vec{K}=\begin{pmatrix}\vec{K}_R & 0\\ 0 & \vec{K}_L\end{pmatrix}, \tag{79}$$

with $\vec{J}_L=\vec{J}_R$, $\vec{K}_L=-\vec{K}_R$ given by Eq. (43), whereas i-parity is represented by the operator

$$\Pi=\begin{pmatrix}0 & \mathbf{1}\\ \mathbf{1} & 0\end{pmatrix}, \tag{80}$$

and the corresponding equation of motion is

$$[B^{j_1j_2}(\vec{p})^2\Pi-\varrho]\ \phi_{RL}(\vec{p})=0\,, \tag{81}$$

which in general is an $\mathcal{O}(p^{2j_{max}})$ equation, with $j_{max}=j_1+j_2$ the maximum value of the total angular momentum. In particular for $j_1=1$ $j_2=\frac{1}{2}$ we obtain an $\mathcal{O}(p^3)$ equation. We have studied many other possibilities for spin $\frac{3}{2}$ following this strategy. All of them yield $\mathcal{O}(p^3)$ equations.

Summarizing up to this point, although a PI can be formulated for the representations obtained as direct product of the simplest representations $(j,0)$ and $(0,j')$, the order of the corresponding eom's exhibit a clear pattern. The higher the spin, the higher the order of the corresponding eom is and for $s>1$ we obtain eom's $\mathcal{O}(p^3)$ or higher order in p. A complete search for the possible eom's that a field in a specific representation can satisfy requires to change our strategy. Indeed, using the PI we obtained equations of motion which the corresponding free fields must necessarily satisfy. However, it is still possible that these free fields satisfy a different eom also. In the next section we use the information on the specific representation (the explicit form of the generators of the HLG) in a different way and explore the possibilities for the existence of linear (Dirac-like) eom for that representation.

IV. BACK TO THE BASICS: COVARIANCE AND LINEAR EQUATIONS.

The order of the equation of motion for a field containing spin j, as dictated by the PI, can be understood from the algebra which, in addition to the Lie algebra, the generators of rotations satisfy. This additional algebra is different for different values of j. Linearity of the eom for spin $\frac{1}{2}$ comes from the fact that generators in this case satisfy the *bilinear* algebra

$$\{J_i, J_j\} = \frac{1}{2}\delta ij \tag{82}$$

whereas the order p^2 equation of motion for particles with $s = 1$ (either Weinberg, Proca or A-K) comes from the *trilinear* algebra satisfied by the $s = 1$ generators

$$J_i J_j J_k + J_k J_j J_i = \delta_{ij} J_k + \delta_{jk} J_i, \tag{83}$$

and the order p^3 of the eom's containing spin $\frac{3}{2}$ comes from the *cuatrilinear* algebra which the $s = \frac{3}{2}$ generators fulfill

$$\{J_{ij}, J_{kl}\} = 5(\delta_{ij} J_{kl} + \delta_{kl} J_{ij}) - \frac{9}{2}\delta_{ij}\delta_{kl}, \tag{84}$$

with $J_{ij} \equiv \{J_i, J_j\}$.

The way out this pattern is the one followed by the KDP equation. This is an $\mathcal{O}(p)$ equation for $s = 1$ fields. If we are able to rederive this equation from group theoretical arguments we will be on the road toward the construction of linear equations for higher spin fields.

The information on the representation we are working with is contained in the generators which can be explicitly constructed for a given representation. We can use this information and the constraints arising from Lorentz covariance to check if there exists or not a linear eom for a given representation. In this way we assume that the field ψ in the given representation satisfy

$$[\beta_\mu p^\mu - m]\,\psi = 0. \tag{85}$$

Lorentz covariance requires β_μ to satisfy

$$[M_{\mu\nu}, \beta_\alpha] = i(g_{\nu\alpha}\beta_\mu - g_{\mu\alpha}\beta_\nu)\,, \tag{86}$$

which in terms of rotations and boosts generators $J_i = \epsilon_{ijk} M^{jk}$ and $K_i = M_{0i}$ read

$$[J_i, \beta_0] = 0,\ [J_i, \beta_j] = i\epsilon_{ijk}\beta_k,\ [iK_i, \beta_0] = -\beta_i,\ [iK_i, \beta_j] = -\delta_{ij}\beta_0. \tag{87}$$

We use these relations to explicitly construct the matrices β_μ.

A. $(j,0)\oplus(0,j)$: fields with single spin

The generators for the $(j,0)\oplus(0,j)$ representation are

$$\vec{J} = \begin{pmatrix} \vec{J}_R & 0 \\ 0 & J_L \end{pmatrix}, \qquad i\vec{K} = \begin{pmatrix} \vec{J}_R & 0 \\ 0 & -J_L \end{pmatrix}, \tag{88}$$

where $\vec{J}_R = \vec{J}_L$ stands for the generators of rotations for a spin j system. Let us write β_0 in the block-matrix form

$$\beta_0 = \begin{pmatrix} b^0_{11} & b^0_{12} \\ b^0_{21} & b^0_{22} \end{pmatrix}, \tag{89}$$

where b^0_{ij} are 2×2 matrices. From the first of relations (87) we obtain

$$\left[\vec{J}^R, b^0_{ij}\right] = 0. \tag{90}$$

Since $\vec{J}^R$ span an irrep of $SU(2)$, by Schur's lemma the b^0_{ij} sub-matrices must be proportional to the identity $b^0_{ij} = b_{ij}\mathbf{1}$, where b_{ij} are numbers. Next we define β_i by the third of relations (87)

$$\beta_i \equiv [-iK_i, \beta_0] = 2\begin{pmatrix} 0 & b_{12}J^R_i \\ -b_{21}J^R_i & 0 \end{pmatrix}. \tag{91}$$

The second of relations (87) is satisfied by this matrix and an explicit calculation of the commutator in the fourth of relations (87) yields

$$[K_i, \beta_j] = -2i\begin{pmatrix} 0 & b_{12}\{J^R_i, J^R_j\} \\ b_{21}\{J^R_i, J^R_j\} & 0 \end{pmatrix}, \tag{92}$$

and the fourth of relations(87) requires $b_{11} = b_{22} = 0$ and

$$\{J^R_i, J^R_j\} = \frac{1}{2}\delta_{ij} \tag{93}$$

The only representation $(j,0)\oplus(j,0)$ whose generators satisfy this relation is $j=\frac{1}{2}$. For higher spin the only solution is the trivial one. Thus, we have shown that there exists no Dirac-like equation of motion for fields transforming in the $(j,0)\oplus(0,j)$ representation for $j>\frac{1}{2}$. Notice in pass that the most general form of Dirac matrices which is consistent with Lorentz covariance only, contains two arbitrary parameters

$$\beta_0 = \begin{pmatrix} \mathbf{0} & a\mathbf{1} \\ b\mathbf{1} & \mathbf{0} \end{pmatrix}, \quad \beta_i = \begin{pmatrix} \mathbf{0} & a\sigma_i \\ -b\sigma_i & \mathbf{0} \end{pmatrix}, \tag{94}$$

where $a\equiv b_{12},\ b\equiv b_{21}$. These two free parameters are just a consequence of the inequivalence of the $(j,0)$ and the $(0,j)$ representation which are separately irreps of the HLG. If we require also invariance under parity we obtain $a=b$. If we further require that the equation describe a particle (anti-particle) of mass m we need $a=1$ ($a=-1$). Under this circumstance the matrices β_μ satisfy the Dirac algebra $\{\beta_\mu,\beta_\nu\} = 2g_{\mu\nu}$ which looks like the "covariantized"version of the algebra satisfied by $\frac{\vec{J}}{s}$ (see Eq.(82)).

B. $(\frac{1}{2},\frac{1}{2})$: Proca representation

Similar calculations for this representation shows that there exists no Dirac-like equation in this case.

C. $(1,0)\oplus(\frac{1}{2},\frac{1}{2})\oplus(0,1)$ again: KDP Equation.

The quantum states for KDP representation in the TPB and TAMB

$$\phi_{TPB} = \; l.\ c. \begin{pmatrix} |1m_1\rangle_R \\ |\frac{1}{2}m\rangle_R \otimes |\frac{1}{2}m'\rangle_L \\ |1m_1\rangle_L \end{pmatrix} , \qquad \phi_{TAMB} = \; l.\ c. \begin{pmatrix} |1m_1\rangle_R \\ |0,0\rangle \\ |1,m\rangle \\ |1m_1\rangle_L \end{pmatrix} , \tag{95}$$

are related by the unitary matrix $\mathcal{M}$ as: $\phi_{TAMB} = \mathcal{M}_{cb}\phi_{TPB}$ with

$$\mathcal{M}_{cb} = \begin{pmatrix} \mathbf{1}_{3\times3} & & \\ & M_{cb} & \\ & & \mathbf{1}_{3\times3} \end{pmatrix} , \tag{96}$$

where M_{cb} is given in Eq.(55). Under this change of basis the generators transform as

$$\vec{J}_{TPB} = \begin{pmatrix} \vec{L} & & \\ & \vec{J}^{(\frac{1}{2},\frac{1}{2})}_{TPB} & \\ & & \vec{L} \end{pmatrix} \longrightarrow \vec{J}_{TAMB} = \begin{pmatrix} \vec{L} & & & \\ & \vec{0} & \vec{0}^\dagger & \\ & \vec{0} & \vec{L} & \\ & & & \vec{L} \end{pmatrix} , \tag{97}$$

where $\vec{J}^{(\frac{1}{2},\frac{1}{2})}_{TPB} = \frac{1}{2}(\vec{\sigma}\otimes\mathbf{1} + \mathbf{1}\otimes\vec{\sigma})$

$$i\vec{K}_{TPB} = \begin{pmatrix} \vec{L} & & \\ & i\vec{K}^{(\frac{1}{2},\frac{1}{2})}_{TPB} & \\ & & -\vec{L} \end{pmatrix} \longrightarrow i\vec{K}_{TAMB} = \begin{pmatrix} \vec{L} & & & \\ & \vec{0} & \vec{B}^\dagger & \\ & \vec{B} & \vec{0} & \\ & & & -\vec{L} \end{pmatrix} \tag{98}$$

with $i\vec{K}^{(\frac{1}{2},\frac{1}{2})}_{TPB} = \frac{1}{2}(\vec{\sigma}\otimes\mathbf{1} - \mathbf{1}\otimes\vec{\sigma})$ and $\vec{B}$ is given in Eq.(58). After a straightforward calculation we obtain

$$\beta_0 = \begin{pmatrix} 0 & 0 & b_{13} & 0 \\ 0 & 0 & 0 & 0 \\ b_{31} & 0 & 0 & b_{34} \\ 0 & 0 & b_{43} & 0 \end{pmatrix} , \quad \vec{\beta} = \begin{pmatrix} \vec{0} & b_{13}\vec{B} & -b_{13}\vec{L} & \vec{0} \\ -b_{31}\vec{B}^\dagger & \vec{0} & \vec{0} & -b_{34}\vec{B}^\dagger \\ b_{31}\vec{L} & 0 & 0 & -b_{34}\vec{L} \\ 0 & b_{43}\vec{B} & b_{43}\vec{L} & 0 \end{pmatrix} , \tag{99}$$

such that

$$(\beta^\mu p_\mu - m)\phi_{TAMB} = 0\,, \tag{100}$$

is covariant. Notice that, similarly to the Dirac case, the most general equation consistent with Lorentz covariance only, contains four arbitrary parameters. Again this is just a consequence of the *four* inequivalent irreps of the HLG contained in KDP representation. The operator for i-parity can be directly obtained from the representation itself as

$$\Pi = \eta \begin{pmatrix} 0 & 0 & 0 & \mathbf{1}_{3\times 3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\mathbf{1}_{3\times 3} & 0 \\ \mathbf{1}_{3\times 3} & 0 & 0 & 0 \end{pmatrix}. \tag{101}$$

If we impose invariance under parity the four free parameters are related as: $b_{13} = -b_{43}$ and $b_{31} = -b_{34}$. Thus, we are left with two free parameters which can be reduced to a $\pm\frac{1}{\sqrt{2}}$ factor if we require that all the fields contained in this equation have the same mass m. For these values of the parameters the matrices β_μ satisfy Kemmer algebra:

$$\beta^\mu\beta^\nu\beta^\rho + \beta^\rho\beta^\nu\beta^\mu = g^{\mu\nu}\beta^\rho + g^{\nu\rho}\beta^\mu. \tag{102}$$

It is worth to remark that this algebra is just the "covariantized" version of the algebra satisfied by $\frac{\vec{J}}{s}$ (see Eq.(83)).

The formal relation to Proca equation can be established if the components of the KDP field are related to each other in a specific way. Indeed taking

$$\phi_{TPB} = \begin{pmatrix} \vec{E} + i\vec{B} \\ A^0 \\ \vec{A} \\ \vec{E} - i\vec{B} \end{pmatrix}, \tag{103}$$

where $\vec{E}^i \equiv G^{0i}$ $B^i \equiv \epsilon^{ijk}G_{jk}$ and the usual definition for the strength tensor $G_{\mu\nu}$, it can be easily shown that A_μ satisfy Proca equation. In other words, equivalence of the Proca and KDP equations require to use a very restricted class of fields in $(1,0)\oplus(\frac{1}{2},\frac{1}{2})\oplus(0,1)$, those fields in $(1,0)\oplus(0,1)$ constructed from $p^\mu A^\nu - p^\nu A^\mu$.

D. Dirac-like Equation for the $\left[(1,0)\otimes(0,\frac{1}{2})\right]\oplus\left[(0,1)\otimes(\frac{1}{2},0)\right]$ representation.

Let us denote through this subsection $\left[(1,0)\otimes(0,\frac{1}{2})\right]$ as "left" and $\left[(0,1)\otimes(\frac{1}{2},0)\right]$ as "right" representations. The generators for the "right" representation can be read from Eq. (43) for the case $j_1 = 1, \quad j_2 = \frac{1}{2}$ as

$$\vec{J}_R^{TPB} = \vec{L}\otimes\mathbf{1} + \mathbf{1}\otimes\vec{S}, \quad i\vec{K}_R^{TPB} = \vec{L}\otimes\mathbf{1} - \mathbf{1}\otimes\vec{S}, \tag{104}$$

where we used the conventional $\vec{L}, \vec{S}$ notation for the spin 1 and spin$\frac{1}{2}$ generators respectively. Now we transform everything from the tensor product basis $|1, m_l\rangle_r \otimes |\frac{1}{2}, m_s\rangle_l$ [7] to the TAMB $|1\frac{1}{2}; jm\rangle$. Schematically

[7] The labels r, l remind us that the corresponding spinors belong to the $(1,0)$ and $(0,\frac{1}{2})$ respectively. This distinction is necessary since these spinors differ from spinors belonging to the $(0,1)$ and $(\frac{1}{2},0)$ in their transformation properties under boosts.

$$|1\frac{1}{2};jm\rangle = U_R\ |1,m_l\rangle_r \otimes |\frac{1}{2}m_s\rangle_l \tag{105}$$

where U_R is the matrix of Clebsch-Gordon coefficients

$$U_R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{1}{3}} & -\sqrt{\frac{2}{3}} & 0 \end{pmatrix}. \tag{106}$$

The Lorentz group generators are transformed accordingly to

$$\vec{J}_R^{TAMB} = U_R J^{\vec{TPB}}{}_R U_R^\dagger = \begin{pmatrix} \vec{\mathcal{J}} & \mathbf{0}_{4\times 2} \\ \mathbf{0}_{2\times 4} & \vec{S} \end{pmatrix}, \quad i\vec{K}_R^{TAMB} = U_R i\vec{K}_R^{TPB} U_R^\dagger = \begin{pmatrix} \frac{1}{3}\vec{\mathcal{J}} & \frac{2}{3}\vec{B}^\dagger \\ \frac{2}{3}\vec{B} & \frac{5}{3}\vec{S} \end{pmatrix}, \tag{107}$$

where $\vec{\mathcal{J}}$ stands for spin$\frac{3}{2}$ generators and

$$B_1^\dagger \equiv \begin{pmatrix} -\sqrt{\frac{3}{2}} & 0 \\ 0 & -\sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & 0 \\ 0 & \sqrt{\frac{3}{2}} \end{pmatrix}, \quad B_2^\dagger \equiv i \begin{pmatrix} \sqrt{\frac{3}{2}} & 0 \\ 0 & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & 0 \\ 0 & \sqrt{\frac{3}{2}} \end{pmatrix}, \quad B_3^\dagger \equiv \begin{pmatrix} 0 & 0 \\ \sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}. \tag{108}$$

The matrices $\vec{B}$ were obtained firstly in Ref. [22] by a different procedure and satisfy

$$\begin{aligned} \mathcal{J}_i\mathcal{J}_j + B_i^\dagger B_j &= i\tfrac{3}{2}\epsilon_{ijk}\mathcal{J}_k + \tfrac{9}{4}\delta_{ij}, & S_iS_j + B_iB_j^\dagger &= -i\tfrac{3}{2}\epsilon_{ijk}S_k + \tfrac{9}{4}\delta_{ij}\,, \\ B_i\mathcal{J}_j - \mathcal{J}_jB_i &= \tfrac{5}{2}i\epsilon_{ijk}B_k\,, & S_iB_j - S_jB_i &= -\tfrac{1}{2}i\epsilon_{ijk}B_k\,, \\ B_i\mathcal{J}_j - S_iB_j &= \tfrac{3}{2}i\epsilon_{ijk}B_k\,, & B_i\mathcal{J}_j - S_jB_i &= i\epsilon_{ijk}B_k\,, \\ B_i^\dagger B_j - B_j^\dagger B_i &= 2i\epsilon_{ijk}\mathcal{J}_k\,, & B_iB_j^\dagger - B_jB_i^\dagger &= -4i\epsilon_{ijk}S_k\,. \end{aligned} \tag{109}$$

Let us now study the "left" representation. As discussed in the previous section, under parity $(1,0)\otimes(0,\frac{1}{2}) \to (0,1)\otimes(\frac{1}{2},0)$, hence parity has a simple representation in the TPB for the whole $\Big[(1,0)\otimes(0,\frac{1}{2})\Big] \oplus \Big[(0,1)\otimes(\frac{1}{2},0)\Big]$. The generators of Lorentz transformations in the TPB can be read from Eqs. (77,78)

$$\vec{J}_L^{TPB} = \vec{L}\otimes\mathbf{1} + \mathbf{1}\otimes\vec{S}, \quad i\vec{K}_L^{TPB} = -\vec{L}\otimes\mathbf{1} + \mathbf{1}\otimes\vec{S}. \tag{110}$$

Next we transform states to the TAMB

$$|1\frac{1}{2};jm\rangle_L = U_L\ |1m_l\rangle_l \otimes |\frac{1}{2}m_s\rangle_r\,, \tag{111}$$

where U_L is the matrix of the corresponding Clebsch-Gordon coefficients which has exactly the same form as U_R in Eq.(106). Transforming the Lorentz group generators for the left representation accordingly we obtain the simple result

$$\vec{J}_L^{TAMB} \equiv U_L \vec{J}_L U_L^\dagger = \vec{J}_R^{TAMB}, \quad \vec{K}_L^{TAMB} \equiv U_L \vec{K}_L U_L^\dagger = -\vec{K}_R^{TAMB}. \tag{112}$$

Finally, the change of the basis from the TPB to the TAMB for the complete $\left[(1,0)\otimes(0,\frac{1}{2})\right] \oplus \left[(0,1)\otimes(\frac{1}{2},0)\right]$ representation is accomplished by the unitary transformation

$$U = \begin{pmatrix} U_R & \mathbf{0} \\ \mathbf{0} & U_L \end{pmatrix}. \tag{113}$$

The generators for rotations and boosts for the complete $\left[(1,0)\otimes(0,\frac{1}{2})\right] \oplus \left[(0,1)\otimes(\frac{1}{2},0)\right]$ representation transform to

$$\vec{J}^{TAMB} = \begin{pmatrix} \vec{J}_R^{TAMB} & \mathbf{0} \\ \mathbf{0} & \vec{J}_L^{TAMB} \end{pmatrix}, \quad \vec{K}^{TAMB} = \begin{pmatrix} \vec{K}_R^{TAMB} & \mathbf{0} \\ \mathbf{0} & \vec{K}_L^{TAMB} \end{pmatrix}, \tag{114}$$

where $\vec{J}_R^{TAMB} = \vec{J}_L^{TAMB}$ and $\vec{K}_L^{TAMB} = -\vec{K}_R^{TAMB}$ are given in Eq.(107); whereas i-parity operator remains invariant

$$\Pi^{TAMB} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}. \tag{115}$$

We have now all what we need to construct the linear equation of motion for fields in the $\left[(1,0)\otimes(0,\frac{1}{2})\right] \oplus \left[(0,1)\otimes(\frac{1}{2},0)\right]$ representation which reads

$$(\beta^\mu p_\mu \ - \ m)\, \Phi(\vec{p}) = 0. \tag{116}$$

We now exploit constraints arising from Lorentz covariance in Eq.(87). Firstly we write β_0 in the block-matrix form

$$\beta_0 = \begin{pmatrix} b^0_{11} & b^0_{12} \\ b^0_{21} & b^0_{22} \end{pmatrix}, \tag{117}$$

where b^0_{ij} are 6×6 matrices. From the first of relations (87) we obtain

$$\left[\vec{J}_R^{TAMB}, b^0_{ij}\right] = 0. \tag{118}$$

The irreducibility of the generators $\vec{\mathcal{J}}, \vec{S}$ in Eq.(107), severely restrict the most general form of the b^0_{ij} matrices. Indeed, using Eq.(118) Schur's lemma requires these matrices to have the form

$$b^0_{11} = \begin{pmatrix} a_{11}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & a_{22}\mathbf{1} \end{pmatrix}, b^0_{12} = \begin{pmatrix} a_{13}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & a_{24}\mathbf{1} \end{pmatrix}, b^0_{21} = \begin{pmatrix} a_{31}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & a_{42}\mathbf{1} \end{pmatrix}, b^0_{22} = \begin{pmatrix} a_{33}\mathbf{1} & \mathbf{0} \\ \mathbf{0} & a_{44}\mathbf{1} \end{pmatrix}, \tag{119}$$

where a_{ij} are constant. The third of relations (87) can be taken as the definition of β_i and with this choice the second of relations (87) is automatically satisfied whereas the remaining commutator, when using relations (109), requires

$$a_{11} = a_{22} = a_{33} = a_{44} = 0, \quad a_{24} = -\frac{1}{2}a_{13}, \quad a_{42} = -\frac{1}{2}a_{31}. \tag{120}$$

At the end the most general form for of the matrices β_μ consistent with Lorentz covariance depend on two free parameters and have the following structure

$$\beta^0 = \begin{pmatrix} \mathbf{0} & a\ A_0 \\ c\ A_0 & \mathbf{0} \end{pmatrix}, \quad \beta^i = \frac{1}{3}\begin{pmatrix} \mathbf{0} & -a\ A_i \\ c\ A_i & \mathbf{0} \end{pmatrix}, \tag{121}$$

where $a \equiv a_{13}, c \equiv a_{31}$ are arbitrary constant and

$$A^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\frac{1}{2}\mathbf{1} \end{pmatrix}, \quad A^i = \begin{pmatrix} 2\mathcal{J}_i & B_i^\dagger \\ B_i & -5S_i \end{pmatrix}. \tag{122}$$

An analysis of this equation in the rest frame shows that it describes a multiplet composed of a spin $\frac{3}{2}$ and a spin $\frac{1}{2}$ particle with masses m and $\frac{m}{2}$ respectively. This equation was also obtained in Ref. [19] and firstly in Ref. [16]. The matrices β_μ do not satisfy the single mass condition [17,19] for Dirac-like equations

$$(\beta \cdot p)^n = p^2(\beta \cdot p)^{n-2} \quad \text{for some integer} \quad n. \tag{123}$$

It is worth to remark that as a consequence of the Clifford algebra satisfied by the Dirac matrices, relation (123) holds in the case of Dirac for $n = 2$, whereas the Kemmer algebra satisfied by the β_μ matrices in KDP theory ensure that the same relation holds, but in this case with $n = 3$. Furthermore these algebras are just the "covariant" version of the algebras satisfied by the operators $\frac{\vec{J}}{s}$. A generalization of these results to the case of spin $\frac{3}{2}$ points to the irreps of the covariant version of the spin $\frac{3}{2}$ cuatrilinear algebra satisfied by $\frac{\vec{J}}{s}$ (see Eq.(32)), namely

$$\{\beta_{\mu\nu}, \beta_{\alpha\beta}\} = \frac{20}{9}(g_{\mu\nu}\beta_{\alpha\beta} + g_{\alpha\beta}\beta_{\mu\nu}) - \frac{8}{9}g_{\mu\nu}g_{\alpha\beta}. \tag{124}$$

where $\beta_{\mu\nu} \equiv \{\beta_\mu, \beta_\nu\}$. The possibility of a Dirac-like equation for fields with spin $\frac{3}{2}$ such that the corresponding matrices β_μ satisfy relation (123) for $n = 4$ and also the cuatrilinear algebra in Eq. (124) is presently under investigation.

V. SUMMARY AND PERSPECTIVES.

In this work we briefly reviewed the derivation of Dirac and Weinberg equations using the "principle of indistinguishability" and explored its consequences for other irreducible representations of the Homogeneous Lorentz Group containing spin ≥ 1. We obtained the following results: i) For the representation $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$ this principle yields a second order equation. The latter equation was originally obtained by Ahluwalia-Kirchbach (A-K) [18]. ii) The $(1, 0) \oplus (\frac{1}{2}, \frac{1}{2}) \oplus (0, 1)$ representation obeys a second order equation. The corresponding field can be decomposed into the direct sum of two fields with a common mass, one of them obeys the $j = 1$ Weinberg equation and the other field satisfy the A-K equation. iii) An $\mathcal{O}(p^3)$ equation is derived from this principle for the $(\frac{1}{2}, \frac{1}{2}) \otimes [(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})]$ representation. iv) All the explored representations containing spin $\frac{3}{2}$ (and lower spins) yield also $\mathcal{O}(p^3)$ equations.

Changing our strategy we used the information on the representations in a different way. Exploiting the specific form of the generators of the HLG for a given representation we explored the existence of first order equations for different representations. We showed that there is no Dirac-like equation for $(j,0)\oplus(0,j)$ fields, except for spin $\frac{1}{2}$. The corresponding matrices β_μ satisfy a Clifford algebra (the covariant version of the *bilinear* algebra satisfied by $\frac{\vec{J}}{s}$). We also concluded that there is no linear equation for the $(\frac{1}{2},\frac{1}{2})$ representation. As for the representation $(1,0)\oplus(\frac{1}{2},\frac{1}{2})\oplus(0,1)$, it satisfies Kemmer-Duffin-Petieau equation. The corresponding matrices β_μ satisfy Kemmer algebra which is just the covariant version of the *trilinear* algebra satisfied by the generators of rotations for $j=1$.

There exists a Dirac-like equation for the $(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ representation which describes a multiplet of two particles, one with spin $\frac{3}{2}$ and mass m and another with spin $\frac{1}{2}$ and mass $\frac{m}{2}$. Based on the single mass condition for Dirac-like equations and on the results previously described we conjectured existence of a Dirac-like equation for spin $\frac{3}{2}$ fields such that the corresponding matrices satisfy the single mass condition Eq.(123) for $n=4$ and the cuatrilinear algebra in Eq.(124).

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